Indian Statistical Institute, Bangalore
M. Math.
Second Year, Second Semester
Operator Theory

Final Examination — Date : May 6, 2011
Maximum marks: 100 — Time: 3 hours

In the following X, Y, Z are complex Banach spaces. For any bounded sub-interval $[a, b]$ of real line, $C[a, b]$ is the Banach space of complex valued continuous functions on $[a, b]$ with respect to the sup-norm

1. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{K}$ be a subspace of $\mathcal{H}$ and let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. Let $N : \mathcal{H} \to \mathcal{H}$ be a bounded linear map. Show that N leaves $\mathcal{K}$ invariant (that is, $N(\mathcal{K}) \subseteq \mathcal{K}$) if and only if $NP = PNP$. If N leaves $\mathcal{K}$ invariant and N is self-adjoint show that $\mathcal{K}$ reduces N (that is, $\mathcal{K}^{\perp}$ is also invariant for N.). [20]

2. Consider the Hilbert space of square summable complex valued functions on integers:

$$\mathcal{L} = \{f : \sum_{n \in \mathcal{Z}} |f(n)|^2 < \infty\}$$

with respect to the inner product $\langle f, g \rangle = \sum_{n \in \mathcal{Z}} \overline{f(n)} g(n)$. Define $S : \mathcal{L} \to \mathcal{L}$ by $Sf(n) = \frac{1}{n^2+1} f(n)$. Show that S is a bounded self-adjoint operator. Obtain its spectral decomposition. [20]

3. Suppose $T : X \to Y$ is a finite rank operator show that T is compact. [10]

4. Let $h : [1, 3] \times [1, 3] \to \mathcal{C}$ be a continuous function. Show that $M : C[1, 3] \to C[1, 3]$ defined by

$$(Mf)(x) = \int_1^3 f(x) h(x, y) dy$$

is a compact operator. (You may state and use Ascoli's theorem.) ([15])

5. Let $J : C[0, 1] \to C[0, 1]$ be defined by

$$Jf(t) = \int_0^t f(s) ds \quad t \in [0, 1].$$

Determine as to whether J is a compact operator or not. [15]

6. Give an example of a compact linear map such that its spectrum is $\{0\}$. [10]

7. Suppose $S : X \to Y$ is a bounded linear map such that $S^* : Y^* \to X^*$ is compact. Show that $S : X \to Y$ is compact. [10]